A GENERALIZATION OF THE ARITHMETIC-GEOMETRIC MEAN INEQUALITY AND AN APPLICATION TO FINITE SEQUENCES OF ZEROS AND ONES

BY

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ABSTRACT

We generalize the arithmetic-geometric mean inequality to a new class of polynomials and give a combinatorial application.

Introduction

The elementary symmetric polynomial $S_r(x_1, \cdots, x_m)$ of degree r has the following property: When $x = (x_1, \cdots, x_m) \geqq 0$ and $\bar{x} = (x_1 + \cdots + x_m)/m$, then $S_r(x_1, \cdots, x_m) \leqq S_r(\bar{x}, \cdots, \bar{x})$. For $r = m$ this is just the arithmetic-geometric mean inequality for $S_m(x_1, \cdots, x_m) = x_1 \cdots x_m \leqq ((x_1 + \cdots + x_m)/m)^m = S_r(\bar{x}, \cdots, \bar{x})$. The generalization to an arbitrary r is due to Mac-Laurin [1]. We prove here a similar inequality for another class of polynomials which occur in a problem of sequences of zeros and ones.

1. The inequality

M denotes always the set of integers $1, \cdots, m$ and 2^M its powerset.

DEFINITION 1.1. When $I = \{i_1, \cdots, i_r\} \in 2^M$ such that $i_1 < \cdots < i_r$ holds, then I is *alternating*, iff i_ν even is equivalent to $i_{\nu+1}$ odd for $\nu = 1, \cdots, r-1$. Then

$$\mathfrak{A}_{r,m} = \{I \in 2^M \colon \ I \ = r, I \text{ alternating}\}. \tag{1.1}$$

$$A_r(x_1, \cdots, x_m) = \Sigma \left\{ \prod_{i \in I} x_i \colon I \in \mathfrak{A}_{r,m} \right\} \tag{1.2}$$

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is the *alternating polynomial* of degree r in m variables. For example

$$\mathfrak{A}_{3,7}: \quad \begin{array}{ccccccc}
1 & 2 & 3 & & & & \\
1 & 2 & & & 5 & & \\
1 & 2 & & & & & 7 \\
1 & & & 4 & 5 & & \\
1 & & & 4 & & & 7 \\
1 & & & & & 6 & 7 \\
 & & 3 & 4 & 5 & & \\
 & & 3 & 4 & & & 7 \\
 & & 3 & & & 6 & 7 \\
 & & & & 5 & 6 & 7 \\
 & 2 & 3 & 4 & & & \\
 & 2 & 3 & & & 6 & \\
 & 2 & & & 5 & 6 & \\
 & & & 4 & 5 & 6 &
\end{array}$$

(The digits of each line are an element of $\mathfrak{A}_{3,7}$).

Let $\beta_{r,m}$ denote the maximum of $A_r(x_1,\cdots,x_m)$ on the simplex

$$S = \{(x_1,\cdots,x_m)\in R^m : x_i \geqq 0 \text{ for } i = 1,\cdots,m,\ \Sigma x_i = 1\}. \tag{1.3}$$

Then we prove the following

THEOREM 1.1. (i) *When* $r \leqq 4$, *then*

$$\beta_{r,m} = \begin{cases} A_r\left(\dfrac{1}{m},\cdots,\dfrac{1}{m}\right) & \text{for } r \equiv m(2) \\ A_r\left(\dfrac{1}{m-1},\cdots,\dfrac{1}{m-1},0\right) & \text{for } r \not\equiv m(2) \end{cases}. \tag{1.4}$$

(ii) *When* $r = 3$ or $r = 4$ *and* $r \equiv m(2)$, *then* $\beta_{r,m}$ *is attained uniquely at* $(1/m,\cdots,1/m)$ *on* S. *When* $r \not\equiv m(2)$, $r \in \{2,3,4\}$, *then* $\beta_{r,m} = \beta_{r,m-1}$ *and* $\beta_{r,m}$ *is not attained at* $(1/m,\cdots,1/m)$.

Whether the condition $r \leqq 4$ may be dropped remains open. When $r = m$, then (1.4) is also true, it is then equivalent to the arithmetic-geometric mean inequality. We prepare the proof be several lemmas, some of them dealing mainly with combinatorial properties of $\mathfrak{A}_{r,m}$.

The *cyclic permutation* T of M is defined by

$$T(i) = \begin{cases} i+1 & \text{for } i < m, \\ 1 & \text{for } i = m. \end{cases} \tag{1.5}$$

In general, when f is a mapping from a set A into a set B, then f develops canonically to a mapping from 2^A into 2^B and we use the same letter for this mapping. So we have for $I \subset M$

$$T(I) = \{T(i): i \in I\}$$

and for $\mathfrak{A} \subset 2^M$

$$T(\mathfrak{A}) = \{T(I): I \in \mathfrak{A}\}.$$

LEMMA 1.1. *$T(\mathfrak{A}_{r,m}) = \mathfrak{A}_{r,m}$ iff $r \equiv m(2)$.*

PROOF. Suppose $I = \{i_1, \cdots, i_r\} \in \mathfrak{A}_{r,m}$ with $i_1 < \cdots < i_r$, so $i_\nu + i_{\nu+1} \equiv 1(2)$ for $\nu = 1, \cdots, r-1$. We have trivially $T(i_\nu) + T(i_{\nu+1}) = i_\nu + i_{\nu+1} + 2 \equiv 1(2)$ for $\nu = 1, \cdots, r-2$ and if $i_r < m$, then also for $\nu = r-1$, proving $T(I) \in \mathfrak{A}_{r,m}$ in this case. When $i_r = m$, then $T(I) \in \mathfrak{A}_{r,m}$ iff $T(m) + T(i_1) \equiv 1(2)$. Now $T(m) + T(i_1) = i_1 + 2 \equiv i_1(2)$ and $i_1 \equiv 1(2)$ iff $i_r \equiv r(2)$. This shows $T(\mathfrak{A}_{r,m}) \subset \mathfrak{A}_{r,m}$ iff $r \equiv m(2)$. From T being one-to-one on the finite set 2^M we obtain $T(\mathfrak{A}_{r,m}) = \mathfrak{A}_{r,m}$. □

To $N \subset M$, $\mathfrak{N} \subset 2^M$ we define

$$\mathfrak{N}[N] = \{I \in \mathfrak{N}: I \cap N = \varnothing\}$$

$$\mathfrak{N}\langle N\rangle = \{U \subset M: U \cap N = \varnothing,\ U \cup N \in \mathfrak{N}\}.$$

When $N = \{n\}$, we write $\mathfrak{N}[n]$, $\mathfrak{N}\langle n\rangle$ instead of $\mathfrak{N}[\{n\}]$, $\mathfrak{N}\langle\{n\}\rangle$.

Then

$$\mathfrak{N} = \mathfrak{N}[n] \,\dot\cup\, \{U \cup \{n\}: U \in \mathfrak{N}\langle n\rangle\}. \tag{1.6}$$

(Here $\dot\cup$ means: union of disjoint sets).

The following lemma is plain:

LEMMA 1.2

$$T(\mathfrak{N}[N]) = (T(\mathfrak{N}))[T(N)],$$

$$T(\mathfrak{N}\langle N\rangle) = (T(\mathfrak{N}))\langle T(N)\rangle.$$

PROOF.

$$\begin{aligned} T(\mathfrak{N}[N]) &= T(\{I \in \mathfrak{N}: I \cap N = \varnothing\}) \\ &= \{J \in T(\mathfrak{N}): T^{-1}(J) \cap N = \varnothing\} \\ &= \{J \in T(\mathfrak{N}): J \cap T(N) = \varnothing\} \\ &= (T(\mathfrak{N}))[T(N)]. \end{aligned}$$

The second equation follows analogously. □

We have from Definition 1.1

$$\begin{cases} \mathfrak{A}_{r,m}[m] = \mathfrak{A}_{r,m-1} \\ \{T^{-1}(I): I \in \mathfrak{A}_{r,m}[1]\} = \mathfrak{A}_{r,m-1} \end{cases} \tag{1.8}$$

and we want to obtain a similar result when $n \neq 1$, m in $\mathfrak{A}_{r,m}[n]$.

DEFINITION 1.2. *When $m > 2$, we define a mapping ϕ_n of M onto $M \setminus \{m, m-1\}$ for every $n \in M$:*

$$\phi_1(j) = \begin{cases} 1 & \text{for } j = 1, m \\ j-1 & \text{for } j = 2, 3, \cdots, m-1 \end{cases}$$

$$\phi_m(j) = \begin{cases} m-2 & \text{for } j = 1, m \\ j-1 & \text{for } j = 2, \cdots, m-1 \end{cases}$$

$$\phi_n(j) = \begin{cases} j & \text{for } j = 1, \cdots, n-1 \\ n-1 & \text{for } j = n \quad (n = 2, \cdots, m-1). \\ j-2 & \text{for } j = n+1, \cdots, m \end{cases}$$

LEMMA 1.3. *When $m \equiv r(2)$ or $1 < n < m$ then*

$$\phi_n(\mathfrak{A}_{r,m}[n]) = \mathfrak{A}_{r,m-2}.$$

PROOF. We first assume $1 < n < m$. Suppose $I \in \mathfrak{A}_{r,m}[n]$.

We observe that $\# \phi_n(I) = r$: Now $\phi_n(a) \neq \phi_n(b)$ for $a \neq b$, iff $\{a, b\} \nsubseteq \{n-1, n, n+1\}$. By hypothesis, $n \notin I$, so in order to show that $\# \phi_n(I) = r$, it suffices to show that $\{n-1, n+1\} \nsubseteq I$. But if $n-1$, $n+1$ were in I they would be successive elements with $(n-1) + (n+1) \equiv 0(2)$ contradicting that I is alternating.

When $u \in M \setminus \{n\}$, then either $\phi_n(u) = u$ or $\phi_n(u) = u - 2$. Therefore, when u, $v \in M \setminus \{n\}$, then $u + v \equiv \phi_n(u) + \phi_n(v)$ (2). This proves $\phi_n(I) \in \mathfrak{A}_{r,m-2}$ and therefore $\phi_n(\mathfrak{A}_{r,m}[n]) \subset \mathfrak{A}_{r,m-2}$.

To prove the reverse inclusion, we define a mapping ψ_n from $M \setminus \{m-1, m\}$ onto $M \setminus \{n, n+1\}$ by

$$\psi_n(j) = \begin{cases} j & \text{for } j < n \\ j+2 & \text{for } j \geqq n \end{cases}$$

When $J \in \mathfrak{A}_{r,m-2}$, then obviously $\psi_n(J) \in \mathfrak{A}_{r,m}[n]$ and $\phi_n(\psi_n(J)) = J$. So $\phi_n(\mathfrak{A}_{r,m}[n]) = \mathfrak{A}_{r,m-2}$.

Now we assume $m \equiv r(2)$ and $n = 1$. We have $\phi_1 \circ T^{-1} = \phi_2$, so

$$\begin{aligned}
&\phi_1(\mathfrak{A}_{r,m}[1]) \\
&= \phi_1(T^{-1}(\mathfrak{A}_{r,m})[T^{-1}(2)]) \qquad \text{(Lemma 1.1)} \\
&= \phi_1(T^{-1}(\mathfrak{A}_{r,m}[2])) \qquad \text{(Lemma 1.2)} \\
&= \phi_2(\mathfrak{A}_{r,m}[2]) \\
&= \mathfrak{A}_{r,m-2}.
\end{aligned}$$

$\phi_m(\mathfrak{A}_{rm}[m]) = \mathfrak{A}_{r,m-2}$ follows analogously. □

The following example gives $\mathfrak{A}_{3,7}[2]$ and $\phi_2(\mathfrak{A}_{3,7}[2])$:

$\mathfrak{A}_{3,7}[2]$	$\phi_2(\mathfrak{A}_{3,7}[2])$
1 4 5	1 2 3
1 4 7	1 2 5
1 6 7	1 4 5
3 4 5	1 2 3
3 4 7	1 2 5
3 6 7	1 4 5
5 6 7	3 4 5
4 5 6	2 3 4

To every element of $\mathfrak{A}_{3,7}[2]$ its image under ϕ_2 is in the same line. Observe that ϕ_2 is not one-to-one.

LEMMA 1.4. *When* $1 < n < m$ then

$$(\mathfrak{A}_{r,m}\langle n-1\rangle)\,[n] = (\mathfrak{A}_{r,m}\langle n+1\rangle)\,[n].$$

PROOF. When $U \in (\mathfrak{A}_{r,m}\langle n-1\rangle)\,[n]$, then $n-1,\ n \notin U$ and $U \cup \{n-1\} \in \mathfrak{A}_{r,m}$. Therefore $n+1 \in U$ is impossible, for otherwise $n-1$ and $n+1$ would be subsequent elements of $U \cup \{n-1\}$ and $(n-1)+(n+1) \equiv 0(2)$. This is a contradiction to $U \cup \{n-1\} \in \mathfrak{A}_{r,m}$. From $n+1 \notin U$ we obtain $U \cup \{n+1\} \in \mathfrak{A}_{r,m}$, too i.e. $U \in (\mathfrak{A}_{r,m}\langle n+1\rangle)\,[n]$. The inverse relation follows analogously. □

Example:

$\mathfrak{A}_{3,7}\langle 1\rangle$	$\mathfrak{A}_{3,7}\langle 3\rangle$	$(\mathfrak{A}_{3,7}\langle 1\rangle)[2] = (\mathfrak{A}_{3,7}\langle 3\rangle)[2]$
2 3	1 2	4 5
2 5	4 5	4 7
2 7	4 7	6 7
4 5	6 7	
4 7	2 4	
6 7	2 6	

COROLLARY. *When* $m \geqq 3$ *and* $r \equiv m(2)$, *then*

$$(1.7) \qquad (\mathfrak{A}_{r,m}\langle T^{-1}(n)\rangle)[n] = (\mathfrak{A}_{r,m}\langle T(n)\rangle)\,[n] \quad \textit{for every } n \in M.$$

Only the cases $n = 1$ and $n = m$ are not covered by the previous lemma.

PROOF.
$$\begin{aligned}
&(\mathfrak{A}_{r,m}\langle T^{-1}(n)\rangle)[n] \\
&= ((T^{n-2}(\mathfrak{A}_{r,m}))\langle T^{-1}(n)\rangle)\,[n] && \text{(Lemma 1.1)}\\
&= ((T^{n-2}(\mathfrak{A}_{r,m}))\langle T^{n-2}(1)\rangle)\,[T^{n-2}(2)] && (1.5)\\
&= (T^{n-2}(\mathfrak{A}_{r,m}\langle 1\rangle))\,[T^{n-2}(2)] && \text{(Lemma 1.2)}\\
&= T^{n-2}((\mathfrak{A}_{r,m}\langle 1\rangle)\,[2]) && \text{(Lemma 1.2)}\\
&= T^{n-2}((\mathfrak{A}_{r,m}\langle 3\rangle)\,[2]) && \text{(Lemma 1.4)}\\
&= (T^{n-2}(\mathfrak{A}_{r,m}\langle 3\rangle))\,[T^{n-2}(2)] && \text{(Lemma 1.2)}\\
&= ((T^{n-2}((\mathfrak{A}_{r,m}))\langle T^{n-2}(3)\rangle)\,[T^{n-2}(2)] && \text{(Lemma 1.2)}\\
&= ((T^{n-2}(\mathfrak{A}_{r,m}))\langle T(n)\rangle)[n] && (1.5)\\
&= (\mathfrak{A}_{r,m}\langle T(n)\rangle)\,[n]. && \text{(Lemma 1.1)} \ \square
\end{aligned}$$

The condition $r \equiv m(2)$ in the corollary is necessary. For example

$$(\mathfrak{A}_{3,6}\langle 6\rangle)[1] = \{\{2,3\},\{2,5\},\{4,5\}\}$$
$$\neq (\mathfrak{A}_{3,6}\langle 2\rangle)[1] = \{\{3,4\},\{3,6\},\{5,6\}\}.$$

The following lemma is obtained analogously to Lemma 1.4 but a little more complicated.

LEMMA 1.5. *When* $m \geqq 4$, $r \geqq 2$ *and* $1 \leqq n \leqq m-3$, *then*

$$(\mathfrak{A}_{r,m}\langle\{n,n+1\}\rangle)[n+2] = (\mathfrak{A}_{r,m}\langle\{n+2,n+3\}\rangle)\,[n+1]$$

PROOF. Suppose $V \in (\mathfrak{A}_{r,m}\langle\{n,n+1\}\rangle)\,[n+2]$. Then $\{n,n+1,n+2\} \cap V = \emptyset$ and $V \cup \{n,n+1\} \in \mathfrak{A}_{r,m}$.

The assumption $n+3 \in V$ gives us again (as in the proof of Lemma 1.4) a contradiction to $V \cup \{n,n+1\} \in \mathfrak{A}_{r,m}$. Therefore $V \cup \{n+2,n+3\} \in \mathfrak{A}_{r,m}$, showing $V \in (\mathfrak{A}_{r,m}\langle\{n+2,n+3\}\rangle)\,[n+1]$. The inverse relation needs no new argument. $\square$

The following corollary follows by a similar calculation as the corollary to Lemma 1.4.

COROLLARY. *When* $m \geqq 4$, $r \geqq 2$, $m \equiv r(2)$ *then we have*

$$(1.8) \qquad (\mathfrak{A}_{r,m}\langle\{n, T(n)\}\rangle)\,[T^2(n)] = (\mathfrak{A}_{r,m}\langle\{T^2(n),\ T^3(n)\}\rangle)\,[T(n)].$$

for every $n \in M$. □

When $I \subset M$, then min I is the smallest element of I. Then we define

$$(1.9) \qquad \begin{cases} a_{r,m} = \#\ \mathfrak{A}_{r,m}, \\ b_{r,m} = \#\{I \in \mathfrak{A}_{r,m}\colon \min I \equiv 1(2)\}, \\ c_{r,m} = \#\{I \in \mathfrak{A}_{r,m}\colon \min I \equiv 0(2)\}. \end{cases}$$

LEMMA 1.6. $c_{r,m} = b_{r,m-1}$ *for* $m > 1$.

PROOF. $T^{-1}(\{I \in \mathfrak{A}_{r,m}\colon \min I \equiv 0(2)\}) = \{J \in \mathfrak{A}_{r,m-1}\colon \min I \equiv 1(2)\}$, therefore

$$\begin{aligned} c_{r,m} &= \#\ (T^{-1}(\{I \in \mathfrak{A}_{r,m}\colon \min I \equiv 0(2)\})) \\ &= \#\ (\{J \in \mathfrak{A}_{r,m-1}\colon \min I \equiv 1(2)\}) \\ &= b_{r,m-1}. \square \end{aligned}$$

LEMMA 1.7. $b_{r,m} = b_{r,m-2} + b_{r-1\ m-1}$ *for* $2 \leqq r \leqq m-2$.

PROOF.

$$\begin{aligned} b_{r\,m} &= \#\ \{I \in \mathfrak{A}_{r,m}\colon \min I = 1\} \\ &\quad + \#\ \{I \in \mathfrak{A}_{r,m}\colon \min I \geqq 3, \min I \equiv 1(2)\} \\ &= \#\ \{V \subset M\colon 1 \notin V,\ V \cup \{1\} \in \mathfrak{A}_{r,m}\} \\ &\quad + \#\ T^{-2}(\{I \in \mathfrak{A}_{r,m}\colon \min I \geqq 3, \min I \equiv 1(2)\}) \\ &= \#\ \{V \in \mathfrak{A}_{r-1,m}\colon \min V \equiv 0(2)\} \\ &\quad + \#\ \{J \in \mathfrak{A}_{r,m-2}\colon \min J \equiv 1(2)\}. \\ &= c_{r-1,m} + b_{r,m-2} \\ &= b_{r-1,m-1} + b_{r,m-2}. \qquad \text{(Lemma 1.6)}\ \square \end{aligned}$$

LEMMA 1.8. $a_{r,m} = \binom{m - [\frac{1}{2}(m-r+1)]}{r} + \binom{m - [\frac{1}{2}(m-r+2)]}{r}$.

(*As usual* $\binom{u}{v} = 0$ *for* $u < v$ *and* $[x]$ *denotes the greatest integer less than or equal to* x).

PROOF. We show by induction:

$$(1.10) \qquad b_{r,m} = \binom{m - [\frac{1}{2}(m-r+1)]}{r}.$$

(1.10) holds for $r = 1$, $r = m-1$, $r = m$. Using Lemma 1.7 and the induction hypothesis we obtain (1.10) and the lemma follows then from (1.10), Lemma 1.6 and $a_{r,m} = b_{r,m} + c_{r,m}$. □

COROLLARY.

$$A_r\left(\frac{1}{m},\cdots,\frac{1}{m}\right) = \frac{1}{m^r}\left[\binom{m-[\frac{1}{2}(m-r+1)]}{r} + \binom{m-[\frac{1}{2}(m-r+2)]}{r}\right].$$

LEMMA 1.9.

$$\left(\frac{1+1/n}{1-1/n}\right)^s < \frac{1+s/n}{1-s/n} \quad \textit{for } 2 \leqq s \leqq n.$$

PROOF. When $s = 1$, equality holds. Now suppose $s > 1$ and we have

$$\left(\frac{1+1/n}{1-1/n}\right)^{s-1} \leqq \frac{1+\dfrac{s-1}{n}}{1-\dfrac{s-1}{n}}$$

by induction hypothesis. Then

$$\left[\frac{1+1/n}{1-1/n}\right]^s \leqq \frac{\left(1+\dfrac{s-1}{n}\right)\left(1+\dfrac{1}{n}\right)}{\left(1-\dfrac{s-1}{n}\right)\left(1-\dfrac{1}{n}\right)}$$

$$= \frac{1+\dfrac{s}{n}+\dfrac{s-1}{n^2}}{1-\dfrac{s}{n}+\dfrac{s-1}{n^2}} < \frac{1+\dfrac{s}{n}}{1-\dfrac{s}{n}}. \quad \square$$

Define

$$C_r(m) = A_r\underbrace{\left(\frac{1}{m},\cdots,\frac{1}{m}\right)}_{m}. \tag{1.11}$$

LEMMA 1.10. *When* $2 < r \leqq m$, $m \equiv r(2)$, *then* $C_r(m) < C_r(m+2)$.

PROOF. We have from $r \equiv m(2)$, the corollary to Lemma 1.8 and (1.11)

$$C_r(m) = \frac{1}{m^r}\left[\binom{\frac{1}{2}(m+r)}{r} + \binom{\frac{1}{2}(m+r-2)}{r}\right].$$

A straightforward but somewhat tedious calculation gives

$$C_r(m+2) - C_r(m) = \frac{1}{2^{r-1}r!}\prod_{v=1}^{r-2}(m+r-2v)\left\{\frac{m+r}{(m+2)^{r-1}} - \frac{m-r+2}{m^{r-1}}\right\}.$$

So all we have to show, that $\{\cdots\}$ is positive. This follows from Lemma 1.9 with $n = m + 1$ and $s = r - 1$. $\square$

We are now prepared to prove our theorem. However for a later application we first state the following

LEMMA 1.11. *When* $2 < r \leqq m$, $m \equiv r(2)$, *then*

$$C_r(m) < \frac{1}{(m+3)^r} \underbrace{A_r(1,\cdots,1,2)}_{m+1}. \quad \square$$

We omit the proof, which is a straightforward but tedious calculation.

PROOF OF THE THEOREM. When $\mathfrak{A} \subset 2^M$, $x \in R^m$ we define the polynomial $P(\mathfrak{A}, \cdot)$ by

$$P(\mathfrak{A}, x) = \Sigma \left\{ \prod_{i \in I} x_i : I \in \mathfrak{A} \right\}. \tag{1.12}$$

(As usual the product over the empty index set is 1). When $\mathfrak{A} = \mathfrak{B} \cup \mathfrak{C}$, then $P(\mathfrak{A}, x) = P(\mathfrak{B}, x) + P(\mathfrak{C}, x)$. Therefore by (1.6)

$$P(\mathfrak{A}, x) = P(\mathfrak{A}[i], x) + x_i P(\mathfrak{A}\langle i \rangle, x).$$

For the alternating polynomial we obtain $A_r(x_1, \cdots, x_m) = P(\mathfrak{A}_{r,m}, x)$.

When $r = 1$, then our theorem is trivial, for $P(\mathfrak{A}_{r,m}, x) \equiv 1$ on S. When $r = 2$, then

$$A_2(x_1, \cdots, x_m) = \left(\sum_{i \equiv 0(2)} x_i \right) \left(\sum_{j \equiv 1(2)} x_j \right)$$

and this product is maximal on S iff

$$\sum_{i \equiv 0(2)} x_i = \sum_{j \equiv 1(2)} x_j. \tag{1.13}$$

So $x_1 = \cdots = x_m = 1/m$ is a solution of (1.13) iff $m \equiv 0(2)$. When $r \not\equiv m(2)$, then $(1/(m-1), \cdots, 1/(m-1), 0)$ is a boundary point of S which solves (1.13). Therefore

$$\underbrace{A_2\left(\frac{1}{m}, \cdots, \frac{1}{m}\right)}_{m} < \beta_{2\,m} = \underbrace{A_2\left(\frac{1}{m-1}, \cdots, \frac{1}{m-1}, 0\right)}_{m}$$

$$= \underbrace{A_2\left(\frac{1}{m-1}, \cdots, \frac{1}{m-1}\right)}_{m-1} = \beta_{2,m-1}.$$

This proves the case $r = 2$ of our theorem.

Now suppose $r \geqq 3$. We build the Lagrange-function

(1.14) $$L(x,\lambda) = P(\mathfrak{A}_{r,m}, x) + \lambda(1 - \langle e, x\rangle).$$

(Here $\langle e,x\rangle$ is the scalar product of $e = (1,\cdots,1)$ and x).

Using (1.6) we have

$$\frac{\partial}{\partial x_{n-1}} P(\mathfrak{A}_{r,m}, x) = P(\mathfrak{A}_{r,m}\langle n-1\rangle, x)$$

$$= P((\mathfrak{A}_{r,m}\langle n-1\rangle)[n], x) + x_n P(\mathfrak{A}_{r,m}\langle\{n-1,n\}\rangle, x),$$

$$\frac{\partial}{\partial x_{n+1}} P(\mathfrak{A}_{r,m}, x) = P(\mathfrak{A}_{r,m}\langle n+1\rangle, x)$$

$$= P((\mathfrak{A}_{r,m}\langle n+1\rangle)[n], x) + x_n P(\mathfrak{A}_{r,m}\langle\{n+1,n\}\rangle, x).$$

For a stationary point $(x_1^*,\cdots,x_m^*, \lambda^*)$ of L we obtain

(1.15) $$\frac{\partial}{\partial x_n} P(\mathfrak{A}_{r,m}, x^*) = \lambda^* \text{ for all } n \in M.$$

The combinatorial preparations serve now to handle the equations (1.15). When $1 < n < m$ we get from Lemma 1.4

$$x_n^* P(\mathfrak{A}_{r,m}\langle\{n-1,n\}\rangle, x^*) = x_n^* P(\mathfrak{A}_{r,m}\langle\{n+1,n\}\rangle, x^*).$$

When x^* is an interior point of S, we have $x_n > 0$. So we obtain for every n with $1 \leqq n < m$

(1.16) $$P(\mathfrak{A}_{r,m}\langle\{1,2\}\rangle, x^*) = P(\mathfrak{A}_{r,m}\langle\{n,n+1\}\rangle, x^*).$$

We have from (1.6) for $2 \leqq n \leqq m-2$ again

$$P(\mathfrak{A}_{r,m}\langle\{n-1,n\}\rangle, x^*)$$
$$= P((\mathfrak{A}_{r,m}\{\langle n-1,n\}\rangle)[n+1], x^*) + x_{n+1}^* P(\mathfrak{A}_{r,m}\langle\{n-1,n,n+1\}\rangle, x^*),$$
$$P(\mathfrak{A}_{r,m} < \{n+1,n+2\} >, x^*)$$
$$= P((\mathfrak{A}_{r,m}\langle\{n+1,n+2\}\rangle)[n], x^*) + x_n^* P(\mathfrak{A}_{r,m}\langle\{n,n+1,n+2\}\rangle, x^*).$$

This gives us together with (1.16) and Lemma 1.5

(1.17) $$x_n^* P(\mathfrak{A}_{r,m}\langle\{n,n+1,n+2\}\rangle, x^*) = x_{n+1}^* P(\mathfrak{A}_{r,m}\langle\{n-1,n,n+1\}\rangle, x^*) .$$

Now we assume $r = 3$. Then $\mathfrak{A}_{3,m}\langle\{n, T(n), T^2(n)\}\rangle = \varnothing$ for every $n \in M$, so $P(\mathfrak{A}_{3,m}\langle\{n, T(n), T^2(n)\}\rangle) = 1$. From (1.17) therefore it follows $x_2^* = \cdots = x_{m-1}^*$. When m odd, then (1.8) gives us $x_1^* = x_2^* = \cdots = x_{m-1}^* = x_m^* = 1/m$.

Returning to the general case we obtain from (1.17) for $r \geqq 3$, $2 \leqq n \leqq m-3$ with $n+1$ instead of n

$$(1.18)\quad x^*_{n+1} P(\mathfrak{A}_{r,m}\langle\{n+1,n+2,n+3\}\rangle, x^*) = x^*_{n+2} P(\mathfrak{A}_{r,m}\langle\{n, n+1, n+2\}\rangle, x^*).$$

It follows from (1.17) and (1.18)

$$x^*_{n+2} x^*_{n+1} P(\mathfrak{A}_{r,m}\langle\{n-1,n,n+1\}\rangle, x^*) = x^*_{n+2} x^*_n P(\mathfrak{A}_{r,m}\langle\{n,n+1,n+2\}\rangle, x^*)$$
$$= x^*_n x^*_{n+1} P(\mathfrak{A}_{r,m}\langle\{n+1,n+2,n+3\}\rangle, x^*).$$

Therefore, when x^* is an interior point of S we obtain

$$(1.19)\quad x^*_{n+2} P(\mathfrak{A}_{r,m}\langle\{n-1,n,n+1\}\rangle, x^*) = x^*_n P(\mathfrak{A}_{r,m}\langle\{n+1,n+2,n+3\}\rangle, x^*).$$

Now suppose $r = 4$. Then

$$\mathfrak{A}_{4,m}\langle\{n-1,n,n+1\}\rangle = \{i \in M: i \equiv n(2),\ i \neq n\}.$$

Therefore (1.19) becomes

$$x^*_{n+2} \Sigma\{x^*_i: i \equiv n(2), i \neq n\} = x^*_n \Sigma\{x^*_i: i \equiv n(2),\ i \neq n+2\} \text{ or equivalently}$$

$$(x^*_{n+2} - x^*_n)(x^*_{n+2} + x^*_n) = (x^*_n - x^*_{n+2})\ (\Sigma\{x^*_i: i \equiv n(2),\ i \neq n, n+2\}).$$

So, when $x^*_{n+2} \neq x^*_n$, then at least one x^*_i; $i \equiv n(2)$ must be negative. Therefore we have reached the conclusion that $x^*_n = x^*_{n+2}$ or more general $x^*_i = x^*_j$ when $2 \leqq i, j \leqq m-1$ and when i,j are both even or both odd. Now suppose m even. Then $r \equiv m(2)$ and so (1.8) gives us $x^*_1 = x^*_3$ and $x^*_{m-2} = x^*_m$, too. With $a = x^*_1$, $b = x^*_2$ we have $x^* = (a, b, a, \cdots, b)$ where $a + b = 2/m$.

In general, when r is even, then $\#\{i \in I: i \text{ odd}\} = \#\{i \in I: i \text{ even}\}$ for every $I \in \mathfrak{A}_{r,m}$. Therefore we have

$$\prod_{i \in I} x^*_i = a^{r/2} \cdot b^{r/2} \leqq \left(\frac{a^{r/2} + b^{r/2}}{2}\right)^2$$

with equality iff $a = b = 1/m$.

We put the results together:

Suppose $r = 3$ and m odd. When (x^*, λ^*) is a stationary point of the Lagrange-function L and x^* is an interior point of S, then $x^* = (1/m, \cdots, 1/m)$.

Suppose $r = 4$, m even and $(x^*_1, x^*_2, \cdots, x^*_{m-1}, x^*_m, \lambda^*) = (x^*, \lambda^*)$ is a stationary point of L with x^* in the interior of S. Then $x^*_1 = x^*_3 = \cdots = x^*_{m-1}$, $x^*_2 = x^*_4 = \cdots = x^*_m$. Furthermore, when y is a point of the interior of S of the form $y = (a, b, \cdots, a, b)$ with $a \neq 1/m$, then $A_r(y) < A_r(1/m, \cdots, 1/m)$.

Now we prove that in these cases (i.e. for $r = 3, 4$, $r \equiv m(2)$)

$$A_r(u_1, \cdots, u_m) < A_r(1/m, \cdots, 1/m)$$

holds when $u=(u_1,\cdots,u_m)$ is a boundary point of S. We apply induction on $m-r$. Indeed, when $m=r$, this inequality is just the arithmetic-geometric mean inequality for $r=m=3,4$. Now suppose $m-r>0$. To every $j\in M$ we define a mapping f_j of $\mathbb{R}^m$ into $\mathbb{R}^{m-2}$ using definition 1.2 as follows: To

$$x=(x_1,\cdots,x_m)\in\mathbb{R}^m,\ k\in\{1,2,\cdots,m\}$$

we define

$$y_k = \Sigma\{x_p\colon p\in M,\ \phi_j(p)=k\}$$

and $f_j(x) = (y_1,\cdots,y_{m-2})$. So for example

$$f_3(x) = (x_1, x_2+x_3+x_4, x_5,\cdots,x_{m-2}).$$

Now suppose $u=(u_1,\cdots,u_{j-1},\ 0,\ u_{j+1},\cdots,u_m)\in S$ is a boundary point of S, so $A_r(u)=P(\mathfrak{A}_{r,m}[j],u)$ and also suppose that $I\in\mathfrak{A}_{r,m}[j]$ and $v=(v_1,\cdots,v_{m-2})$ $=f_j(u)$.

When c and d are different elements of $M\setminus\{j\}$ with $\phi_j(c)=\phi_j(d)$, then $\{c,d\}$ $=\{T^{-1}(j),\ T(j)\}$. We have $\{T^{-1}(j),\ T(j)\}\nsubseteq I$ for every $I\in\mathfrak{A}_{r,m}[j]$, for $T^{-1}(j)$, $T(j)$ would be subsequent elements of I of the same parity. Therefore ϕ_j is one-to-one on every $I\in\mathfrak{A}_{r,m}[j]$, hence $v_{\phi_j(p)}=u_p$ for every $p\in I$ and so trivially

$$\prod_{p\in I} u_p = \prod_{k\in\phi_j(I)} v_k.$$

We define a mapping g on $\{I\in\mathfrak{A}_{r,m}[j]\colon T^{-1}(j)\in I\}$ by $g(I)=(I\setminus\{T^{-1}(j)\})$ $\cup\{T(j)\}$. Then $g(I)\in\mathfrak{A}_{r,m}[j]$, too and g maps $\{I\in\mathfrak{A}_{r,m}[j]\colon T^{-1}(j)\in I\}$ one-to-one onto $\{K\in\mathfrak{A}_{r,m}[j]\colon T(j)\in K\}$. This gives us

$$\prod_{p\in I} u_p + \prod_{s\in g(I)} u_s = (u_{T^{-1}(j)}+u_{T(j)})\ \prod\{u_t\colon t\in I,\ t\neq T^{-1}(j)\} = \prod_{k\in\phi_j(I)} v_k.$$

So finally using Lemma 1.3 we obtain

$$\begin{aligned}
P(\mathfrak{A}_{r,m}[j],u) &= \Sigma\left\{\prod_{k\in\phi_j(I)} v_k\colon I\in\mathfrak{A}_{r,m}[j]\right\}\\
&= \Sigma\left\{\prod_{k\in K} v_k\colon K\in\phi_j(\mathfrak{A}_{r,m}[j])\right\}\\
&= \Sigma\left\{\prod_{k\in K} v_k\colon K\in\mathfrak{A}_{rm-2}\right\}\\
&= P(\mathfrak{A}_{r,m-2},v)=A_{r,m-2}(v_1,\cdots,v_{m-2}).
\end{aligned}$$

Trivially $v_1+\cdots+v_{m-2}=1,\ v_i\geqq 0$ for $i=1,\cdots,m-2$. Therefore the induction hypothesis leads to

$$A_{r\,m-2}(v_1,\cdots,v_{m-2})\leqq A_r\left(\frac{1}{m-2},\cdots,\frac{1}{m-2}\right)=C_r(m-2).$$

From Lemma 1.10 we have therefore

$A_{r,m}(u_1,\cdots,u_m) \leqq C_r(m-2) < C_r(m)$ for every point on the boundary of S.

Therefore $\beta_{r,m}$ is not attained at the boundary of S and is consequently attained at interior points x^* of S only, for which (x^*,λ^*) is a stationary point of L. This proves our theorem in the case $r \equiv m(2)$.

Now suppose $r \not\equiv m(2)$, $r = 3$ or $r = 4$ and $(x_1,\cdots,x_m) \in S$. Then

$$\begin{aligned} A_r(x_1,\cdots,x_m) &= A_r(x_1,\cdots,x_m,0) = P(\mathfrak{A}_{r,m+1},(x_1,\cdots,x_m,0)) \\ &= P(\mathfrak{A}_{r,m+1}[m+1],(x_1,\cdots,x_m,0)) = P(\mathfrak{A}_{r,m-1},f_{m+1}(x_1,\cdots,x_m,0)) \\ &= P(\mathfrak{A}_{r,m},(x_1+x_m,x_2,\cdots,x_{m-1})) = A_r(x_1+x_m,x_2,\cdots,x_{m-1}) \end{aligned}$$

so

$$A_r(x_1,\cdots,x_m) = A_r(x_1+x_m,x_2,\cdots,x_{m-1}) \leqq \beta_{r,m-1}. \tag{1.20}$$

This shows $\beta_{r,m} \leqq \beta_{r,m-1}$. The inverse relation is trivial. We have

$$f_m\Big(\underbrace{\Big(\frac{1}{m},\cdots,\frac{1}{m},0\Big)}_{m+1}\Big) = \underbrace{\Big(\frac{1}{m},\cdots,\frac{1}{m},\frac{2}{m}\Big)}_{m-1}$$

and with $x = (1/m,\cdots,1/m)$ the calculation above gives us therefore

$$\begin{aligned} A_r\Big(\frac{1}{m},\cdots,\frac{1}{m}\Big) &= P\Big(\mathfrak{A}_{r,m+1}, f_m\Big(\frac{1}{m},\cdots,\frac{1}{m},0\Big)\Big) \\ &= P\Big(\mathfrak{A}_{r,m-1}, \underbrace{\Big(\frac{1}{m},\cdots,\frac{1}{m},\frac{2}{m}\Big)}_{m-1}\Big) \\ &< P\Big(\mathfrak{A}_{r,m}, \underbrace{\Big(\frac{1}{m-1},\cdots,\frac{1}{m-1}\Big)}_{m-1}\Big) = \beta_{r,m-1} = \beta_{r,m}. \end{aligned}$$

This completes the proof of the theorem. □

COROLLARY TO THEOREM 1.1 *When $r \in \{2,3,4\}$, then*

$$A_r(x_1,\cdots,x_m) \leqq A_r(\bar{x},\cdots,\bar{x}) \tag{1.21}$$

for all $(x_1,\cdots,x_m)$ with $x_i \geqq 0$ for $i = 1,\cdots,m$ iff $r \equiv m(2)$.

This follows immediately from theorem 1.1(i) and from A_r being homogenous of degree r.

For the alternating polynomial (1.2) one may also use the arithmetic-geometric mean inequality separately for every term. This leads for $x = (x_1, \cdots, x_m) \geqq 0$ to

$$A_r(x_1, \cdots, x_m) \leqq \Sigma \left\{ \left(\sum_{i \in I} \frac{x_i}{r} \right)^r I \in \mathfrak{A}_{r,m} \right\} . \tag{1.22}$$

However, the inequality (1.21) gives in general a sharper estimate. For example when $r = 3$, $m = 5$, $x_i = i$ for $i = 1, \cdots, 5$ then $A_r(1, 2, \cdots, 5) = 120$, the right side of (1.21) is 135, the right side of (1.22) is 155.

2. An application to a combinatorial problem

DEFINITION 2.1. When $d = (d_1, \cdots, d_t)$ is a binary number, then a subsequence of d

$$\delta = (d_{i_1}, \cdots, d_{i_r})$$

is called *alternating*, if $d_{i_\nu} + d_{i_{\nu+1}} = 1$ for $\nu = 1, \cdots, r-1$. Then

$$a_r(d) = \# \{\delta = (d_{i_1}, \cdots, d_{i_1}) \colon \delta \text{ alternating}).$$

When D_t is the set of all binary numbers with t digits we define

$$\alpha_{r,t} = \max \{a_r(d) \colon d \in D_t\}.$$

When $\check{d} = (0, 1, \cdots) \in D_t$ is the binary number which forms an alternating sequence itself and which has 0 as its first digit, then we conjecture:

$$\alpha_{r,t} = a_r(\check{d}) \text{ for every } r = 1, \cdots, t. \tag{2.1}$$

This conjecture is near at hand. However, we were not able to solve it in general.

We pointed out in [2], that (2.1) has a graph theoretical meaning, too. The following definitions 2.2,3 and Lemma 2.1 which prepare theorem 2.1 are essentially the same as in [2].

DEFINITION 2.2. Suppose $M(t) = \{1, 2, \cdots, t\}$, $d \in D_t$, $p, q \in M(t)$, $p \leqq q$. Then p, q are *d-equivalent* iff $d_p = d_{p+1} = \cdots = d_q$. The equivalence classes are the *blocks of d*. When $\delta = (d_{i_1}, \cdots, d_{i_r})$, $\delta^* = (d_{j_1}, \cdots, d_{j_r})$ are subsequences of d, then δ, δ^* are *equivalent*, iff $r = s$ and i_ν, j_ν are d-equivalent for $\nu = 1, \cdots, r$.

It is plain that, when p, q are not d-equivalent with $p < q$ and when q, q^* are d-equivalent, then $p < q^*$, too. Therefore the natural order devolves upon blocks of d.

DEFINITION 2.3. Suppose $d \in D_t$ has the sequence of blocks

$$(N_1, \cdots, N_m), N_1 < \cdots < N_m$$

a) When $\# N_i = n_i$, then

$$\nu(d) = (n_1, \cdots, n_m)$$

is the *partition of d*.

b) f_d is the mapping of $M(t) = \{1, \cdots, t\}$ onto $M = \{1, \cdots, m\}$ defined by

$$i \in N_{f_d(i)} \,.$$

Obviously the partition $\nu(d)$ of d is a partition of t, i.e. is an ordered sequence of positive integers n_i with $n_1 + \cdots + n_m = t$. We write $0' = 1, 1' = 0$ and to $d = (d_1, \cdots, d_m)$ then $d' = (d_1', \cdots, d_m')$. When ν is any partition of t, then there exist exactly two binary numbers d, $d^* \in D_t$ with $\nu(d) = \nu(d^*) = \nu$ and here $d^* = d'$. We have $a_r(d) = a_r(d')$ for every $d \in D_t$. So when $\alpha_{r,t} = a_r(d)$, then $\alpha_{r\,t} = a_r(d')$, too.

LEMMA 2.1. *For every $d \in D_t$ with $\nu(d) = (n_1, \cdots, n_m)$ we have*

$$a_r(d) = A_r(n_1, \cdots, n_m)$$

PROOF. When $\delta = (d_{j_1}, \cdots, d_{j_r})$ is a subsequence of d, then the number of sequences equivalent with δ is

$$n_{f_d(j_1)} \cdot \cdots \cdot n_{f_d(j_r)}.$$

Now δ is alternating iff $f_d(j_\nu) + f_d(j_{\nu+1}) \equiv 1(2)$ for $\nu = 1, \cdots, r-1$, i.e. iff $\{f_d(j_1), \cdots, f_d(j_r)\} \in \mathfrak{A}_{r,m}$. This gives us

$$\begin{aligned} &\{\delta = (d_{i_1}, \cdots, d_{i_r}) \colon \delta \text{ alternating}\} \\ &= \Sigma \left\{ \prod_{\nu=1}^{r} n_{f_d(j_\nu)} \colon \{ f_d(j_1), \ \cdots, \{f_d(j_r)\} \in \mathfrak{A}_{r,m} \right\} \\ &= \Sigma \left\{ \prod_{i \in I} n_i \colon I \in \mathfrak{A}_{r,m} \right\} \\ &= A_r(n_1, \cdots, n_m). \ \square \end{aligned}$$

We use Theorem 1.1 for proving the conjecture (2.1) when $r = 1, 2, 3, 4$.

THEOREM. 2.1. *When $r \leqq 4$, then*

$$\alpha_{r,t} = a_r(\dot{d}).$$

When $r = 3, 4$, then $\alpha_{r,t}$ is attained only *for $\dot{d}$ or $\dot{d}'$ iff $r \equiv t(2)$.*

PROOF. Suppose $d \in D_t$ has the partition $\nu(d) = (n_1, \cdots, n_m)$. We have to regard several cases separately.

1) $r = 1$. This case follows from $a_1(d) = t$ for every $d \in D_t$.

2) $r = 2$. It is

$$\left[\frac{t+1}{2}\right] \cdot \left[\frac{t}{2}\right] = \max\{u \cdot v : u, v \text{ positive integers}, u + v = t\}.$$

With

$$\begin{aligned} u(d) &= \Sigma\{n_i : i \text{ odd}\} \\ g(d) &= \Sigma\{n_j : j \text{ even}\} \end{aligned}$$

we have

$$a_2(d) = u(d) \cdot g(d)$$

and therefore

$$\max\{a_2(d) : d \in D_t\} \leqq \left[\frac{t+1}{2}\right] \cdot \left[\frac{t}{2}\right].$$

It is $u(\bar{d}) = \left[\frac{t+1}{2}\right]$, $g(\bar{d}) = \left[\frac{t}{2}\right]$,

proving the case $r = 2$.

3) $r = 3, 4, t \equiv r(2)$. Then by the corollary to Theorem 1.1:

$$\begin{aligned} a_r(d) &= A_r(n_1, \cdots, n_m) = A_r(n_1, \cdots, n_m, \underbrace{0, \cdots, 0}_{t-m}) \\ &\leqq A_r(\underbrace{1, \cdots, 1}_{t}) = a_r(\bar{d}). \end{aligned}$$

Here equality holds iff $m = t$, equivalently iff $n_1 = \cdots = n_m = 1$ i.e., iff $d = \bar{d}$ or $d = \bar{d}'$.

4) $r = 3, 4, r \not\equiv t(2), m \leqq t - 3$. Then

$$\begin{aligned} a_r(d) &= A_r(n_1, \cdots, n_m) = A_r(n_1, \cdots, n_m, \underbrace{0, \cdots, 0}_{t-m-3}) \\ &\leqq A_r\left(\frac{t}{t-3}, \cdots, \frac{t}{t-3}\right) && \text{(corollary to Theorem 1.1)} \\ &= t^r C_r(t-3) && (1.11) \\ &< A_r(\underbrace{1, \cdots, 1}_{t-1}, 2) && \text{(Lemma 1.11)} \\ &= A_r(\underbrace{1, \cdots, 1}_{t}) && (1.20) \\ &= a_r(\bar{d}). \end{aligned}$$

5) $r = 3, 4$, $r \not\equiv t(2), m = t - 2$. Here $r \not\equiv m(2)$ and therefore with (1.20)

$$a_r(d) = A_r(n_1, \cdots, n_m) = A_r(n_1 + n_m, n_2, \cdots, n_{m-1}).$$

There exists a $\hat{d} \in D_{m-1}$ with $v(\hat{d}) = (n_1 + n_m, n_2, \cdots, n_{m-1})$, so $a_r(d) = a_r(\hat{d})$. From part 4 of this proof we obtain $a_r(\hat{d}) < a_r(\bar{d})$, showing

$$a_r(d) < a_r(\bar{d}).$$

6) $r = 3, 4$, $r \not\equiv t(2)$, $m = t - 1$. Here exists exactly one $j \in M$ with $n_j = 2$ and it is $n_i = 1$ for every $i \in M \backslash \{j\}$, so

$$a_r(d) = A_r(1, \cdots, \underset{j}{2}, \cdots, 1).$$

From $r \equiv m(2)$, Lemma 1.1 and (1.12) immediately follows

$$A_r(1, \cdots, 2, \cdots, 1) = A_r(2, 1, \cdots, 1).$$

From (1.20) we obtain

$$A_r(\underbrace{2, 1, \cdots, 1}_{t-1}) = A_r(\underbrace{1, \cdots, 1}_{t}) = a_r(\bar{d}),$$

so together

$$a_r(d) = a_r(\bar{d}),$$

completing the proof. □

The conjecture 2.1 may also be verified directly when $m - r = 0, 1, 2$.

Addendum. After this paper was sent for publication the author proved Theorem 1.1 without the restriction $r \leqq 4$ and Heiko Harborth proved the conjecture 2.1.

References

1. C. Mac-Laurin, *A second letter to Martin Folges, Esq., concerning the roots of equations with the demonstration of other rules in algebra*, Philos. Trans. **36** (1729), 59–96.

2. F. Hering, *Nested bipartite graphs*, Israel J. Math. **9** (1971), 403–417.

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